Eigenvalues, eigenvectors, and the characteristic polynomial

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1 Introduction

The eigenvalue of a linear transformation (also referred to the characteristic vector) defines a direction that is invariant under transformation. 'Eigen' is adopted from German 'eigen' which means essentially 'unique to.'

Let us define the $n \times n$ matrix A, then the eigenvalue λ is related to A and the eigenvector ν by

$$A\nu = \lambda\nu \tag{1}$$

In other words, there is a vector ν where the direction does not change with the transformation of the vector. The result of the transformation is a scaled multiple of the vector, scaled by λ. The above equation denotes a right eigenvector, due to the order of the multiplication of ν.

The set of all eigenvalues associated with a matrix is referred to as the eigensystem of that matrix. The eigenspace of A is the set of all eigenvectors associated with each eigenvalue (including the zero vector).

The eigenvalue concept has a rich history mathematically going back to Euler in the 18th century. It wasn't until the early 19th century when Cauchy worked out how to apply this to quadric surfaces and generalized it to n-dimensional spaces. Others have expanded the applications and mathematical richness significantly.

This concept has applications in many fields such as dynamical systems and control, strength of materials, image processing, really any area where transformations occur.

2 Computing the eigenvalues and eigenvectors

There are a number of ways to solve for the eigenvalues. Here we discuss the method of computing the solution to the characteristic equation.

Since the eigenvalue equation is given by

$$A\nu = \lambda\nu \tag{2}$$

This can be written

$$(A - \lambda I)\nu = 0 \tag{3}$$

where I is an $n \times n$ identity matrix (ones only on diagonal, all other entries zero). A known result from linear algebra is that the above equation only has a non-zero solution ν if the determinant is zero. Thus the eigenvalues can be found by computing

$$det|A - \lambda I| = 0 \tag{4}$$

This results in a polynomial function for λ of order n, referred to as the characteristic polynomial.

2.1 Example: 3x3 matrix

Let A be given as

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{bmatrix} \tag{5}$$

then we find the characteristic polynomial by computing

$$det(A - \lambda I) = det\left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = det\left(\begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 4 \\ 0 & 4 & 9-\lambda \end{bmatrix}\right) \tag{6}$$

which gives the characteristic polynomial

$$(2 - \lambda)[(3 - \lambda)(9 - \lambda) - 16] = -\lambda^3 + 14\lambda^2 - 35\lambda + 22 = 0 \tag{7}$$

which, when solved, gives λ roots 2, 1, and 11. Plugging these back into the eigenvalue equation, one at a time, and solving for ν gives the eigenvectors $\nu_1 = [1, 0, 0]^T$, $\nu_2 = [0, 2, -1]^T$, and $\nu_3 = [0, 1, 2]^T$

2.2 Computing the eigenvalues in matlab

Use 'eig(A)' to compute the eigenvalues of a square matrix A in matlab or freemat.

Above order 5, the solution must be computed numerically, so this is a good command to know. Also most graphing calculators can calculate eigenvalues for you these days.

2.3 Useful mathematical note

The diagonal elements (along the main diagonal) of a diagonal or triangular matrix are the eigenvalues of that matrix.

3 References

This document is based on a few books, and for further quick reading, the wikipedia page on eigenvalues is pretty good (we follow their outline to some extent here). See below

- `http://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors`
- D. C. Lay. Linear Algebra and Its Applications. Pearson, 4th edition, 2011.